REGIONS OF EFFECTIVE STABILITY IN THE RESTRICTED PROBLEM OF THREE BODIES

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ABSTRACT

We study the spatial restricted problem of three bodies in the light of Nekhoroshev's theory of stability over exponentially large time intervals. We consider in particular the Sun-Jupiter model and the Trojan asteroids in the neighborhood of the Lagrangian points L_4 and L_5. We find regions of effective stability around the points L_4 and L_5 so that if the initial datum of an orbit is inside these regions the orbit is confined in a slightly larger neighborhood of the equilibrium (in phase space) for a very long time interval. By combining analytical methods and numerical approximations we are able to prove that stability over the age of the universe is guaranteed on a realistic region. The best previous result using similar methodology, for the planar restricted problem proved stability over the age of the universe on a region big enough to include a few real asteroids. By comparing this result with the one gained for the spatial problem we see that the regions of stability in the two cases are of the same magnitude.

1. Introduction

The study of a Hamiltonian system in the neighborhood of an elliptic equilibrium point is of interest in many fields of mathematical physics and astronomy. Let us consider an analytic Hamiltonian H with n degrees of freedom, having an elliptic equilibrium point at the origin. Rigorous results proving the existence of orbits which do not leave a neighborhood of the equilibrium can be given in the framework of KAM theory, under generic conditions of non-resonance and non-degeneracy. KAM guarantees the existence of many n-dimensional invariant tori around the origin. However, such invariant tori do not fill an open region, i.e. the possibility of the so-called Arnold diffusion cannot be excluded, except for the two dimensional case. An alternative approach is to look for results which are valid over a finite time interval, but give an effective bound on the Arnold diffusion. This goal can be achieved by constructing the normal form of the Hamiltonian around the origin.

Normal forms are a standard tool in Celestial Mechanics for studying the dynamics in the neighborhood of an elliptic equilibrium point. Usually these normal forms are obtained as divergent series but their asymptotic character makes them useful. Roughly speaking one shows that the system admits a number of approximate integrals the variation of which in time can be controlled to be small for an extremely long time. This is the basis to derive the classical Nekhoroshev estimates (Nekhoroshev, 1977). In these cases we have effective stability, i.e. even when the system is not stable, the time needed for orbits to leave the neighborhood of the equilibrium is larger than the expected lifetime of the studied physical system.

The stability of the Trojan asteroids is a classical example of this kind. A first model for the problem is provided by the two dimensional (2D) planar, and the three dimensional (3D) spatial restricted three body problem (Szebehely, 1967). One has to estimate the rate of diffusion around the elliptic equilibrium point L_4. Because of the symmetries of the system the same study is valid for the L_5 point. The problem has been previously investigated by Giorgilli et al. (1989), Celletti and Giorgilli (1991) and Giorgilli and Skokos (1997) (hereafter paper I).

The estimation of the region of effective stability by Giorgilli et al. (1989) and Celletti and Giorgilli (1991) was realistic but the region where the real asteroids were actually found was larger by a factor 300 (in the best case) to 3,000 compared to the estimated stability region. This estimation was significantly improved in paper I, since the region found in the planar restricted three body problem was big enough to include 4 real asteroids, while most of them fail to be inside this region by a factor 10.

In the present study we follow the same procedure as in paper I working in the spatial case. We numerically compute the normal form up to order 30, which is really a hard task to do since one has to manipulate functions with a huge number of coefficients. Also the expansion of the Hamiltonian of the system in a power series suitable for the application of the normal form scheme is computed with greater accuracy than before.

2. The Hamiltonian and the normal form of the system

The spatial restricted problem of three bodies, in particular for the Sun (S), Jupiter (J) and asteroid (A) case can be described as follows: we study the motion of an asteroid A of infinitesimal mass, orbiting in the gravitational field of two primaries S and J with masses equal to $1-\mu$ and μ respectively, which are assumed to revolve in circular orbits around their common center of mass. Our formalism is similar to the one described in detail in paper I and by Skokos (1997), generalized for three degrees of freedom. For the sake of completeness we recall the main points of this formalism.

We introduce a uniformly rotating frame (O, q_1, q_2, q_3) so that its origin is located at the center of mass of the Sun-Jupiter system, with the Sun always at the point $(\mu, 0, 0)$ and Jupiter at the point $(1-\mu, 0, 0)$. The physical units are chosen so that the distance between Jupiter and the Sun is 1, $\mu=9.5387536\cdot10^{-4}$ and the angular velocity of Jupiter is 1. The time unit is $(2\pi)^{-1}T_J$ where T_J is the period of the circular motion of Jupiter around the Sun. Thus the age of the universe is about 10^{10} time units. The Hamiltonian of the system is:

$$H=\frac{1}{2}(p_1^2+p_2^2+p_3^2)+q_2p_1-q_1p_2-\frac{1-\mu}{\sqrt{(q_1-\mu)^2+q_2^2+q_3^2}}-\frac{\mu}{\sqrt{(q_1+1-\mu)^2+q_2^2+q_3^2}} \quad (1).$$

In order to bring the Hamiltonian in a form suitable for the application of the normal form scheme we perform a sequence of transformations. We introduce a uniformly rotating frame with its origin on the Sun (S) using the generating function $W_3 = -(Q_1+\mu)\,p_1 - Q_2\,p_2 - Q_3\,p_3 + \mu\,Q_2$, where $Q_1, Q_2, Q_3, P_1, P_2, P_3$ are the heliocentric coordinates.

It is known that the projection of the stability region on the plane of Jupiter's orbit is a banana-shaped region which lies close to the circle with center the Sun and radius equal to the Sun-Jupiter distance. Since the plane of Jupiter's orbit is a symmetry plane for the system a good choice for describing this region is the cylindrical coordinates ρ, θ, Z, which are introduced by the generating function $W_3 = -\rho(P_1 \cos\theta + P_2 \sin\theta) - Z P_3$.

Moving now the origin of the coordinate system on the point L_4 by using the canonical transformation generated by $W_2 = p_x(\rho - 1) + (p_y + 1)\theta - 2\pi p_y/3 + p_z Z$, the Hamiltonian becomes:

$$H = \frac{1}{2}\left[p_x^2 + \frac{(p_y+1)^2}{(x+1)^2} + p_z^2\right] - p_y - \mu(x+1)\cos\left(y + \frac{2\pi}{3}\right) - \frac{1-\mu}{\sqrt{(x+1)^2 + z^2}} - \frac{\mu}{\sqrt{(x+1)^2 + z^2 + 1 + 2(x+1)\cos\left(y + \frac{2\pi}{3}\right)}} \quad (2),$$

where x, y, z, p_x, p_y, p_z are the new canonical coordinates.

We expand the Hamiltonian (2) in Taylor series around the point L_4 ($x = y = z = p_x = p_y = p_z = 0$) using the computer program Mathematica. The program allows us to compute the coefficients with accuracy up to 15 decimal digits, while in paper I the corresponding expansion was made by a not so highly accurate program.

The next transformation is performed following paper I. It gives the quadratic part of the Hamiltonian the diagonal form $H_2 = \sum_{j=1}^{3} \omega_j \cdot (x_j^2 + y_j^2)/2$ where $x_1, x_2, x_3, y_1, y_2, y_3$ are the canonical coordinates and $\omega_1 = 9.967575 \cdot 10^{-1}$, $\omega_2 = -8.046388 \cdot 10^{-2}$, $\omega_3 = 1$ are the frequencies. For the entire Hamiltonian we have $H = \sum_{s \geq 2} H_s$ where H_s is a homogeneous polynomial of degree s.

Finally, following Giorgilli et al. (1989) we use the formalism of Lie transforms to construct the normal form $Z^{(r)}$ up to order r:

$$Z^{(r)}(x_1', x_2', x_3', y_1', y_2', y_3') = Z_2 + Z_3 + Z_4 + \ldots + Z_r + Y^{(r)} \quad (3),$$

so that Z_s , s=2,...,r are homogeneous polynomials of degree s which depend only on the actions $I_j' = \frac{1}{2}(x_j'^2 + y_j'^2)$, j=1,2,3 and $Y^{(r)}$ is the remainder, a power series starting with terms of degree r+1. The algorithm for the computation of the normal form is given by Giorgilli (1979).

In paper I where the 2D case (planar problem) was studied the power series of 4 variables were truncated at order r=35. A function of 4 variables with terms up to order 35 requires 82,251 coefficients, while the process of constructing the normal form requires the computation of several functions with a total of 2,549,782 coefficients. In the present 3D case (spatial problem) we use expansions of functions with 6 variables up to order r=30. This is a much harder task compared to the 2D case since a function of 6 variables with terms up to order 30 requires 1,947,792 coefficients and the program which calculates the normal form manipulates 55,929,459 coefficients.

3. Estimation of the effective stability region

The normal form $Z^{(r)}$ admits three approximate first integrals:

$$I_j' = \frac{1}{2}(x_j'^2 + y_j'^2) \;, \quad j = 1, 2, 3 \quad (4).$$

These are not exact integrals and their variation rate is given by:

$$\dot{I}'_j = \left[I'_j , Z^{(r)} \right] \overset{(3)}{=} \left[I'_j , Y^{(r)} \right], j=1,2,3 \qquad (5),$$

which is a power series starting with terms of degree r+1. We remark that [f , g] stands for the Poisson bracket of functions f and g.

We introduce now suitable domains in the phase space where we study the stability properties of the system and also a norm, which allows us to calculate the time variations of the three integrals (4). For fixed positive constants R_1, R_2, R_3 we consider a family of domains of the form:

$$\Delta_{\rho R} = \left\{ (x', y') \in R^6 : x_j'^2 + y_j'^2 \le \rho^2 R_j^2 ,\ j=1,2,3 \right\} \qquad (6),$$

where x stands for x'_1, x'_2, x'_3 and y for y'_1, y'_2, y'_3. For $(x', y') \in \Delta_{\rho R}$ we have $I'_j \le \rho^2 R_j^2/2$ for j=1,2,3. The norm $\|f\|_{\rho R}$ of a homogeneous polynomial $f(x', y')$ of degree s in the domain $\Delta_{\rho R}$ does not exceed the quantity (Skokos, 1997):

$$\|f\|_{\rho R} < \frac{\rho^s}{2^{s/2}} \sum_{j_1 j_2 j_3 k_1 k_2 k_3} | C_{j_1 j_2 j_3 k_1 k_2 k_3} | \, R_1^{j_1+k_1} R_2^{j_2+k_2} R_3^{j_3+k_3} \qquad (7),$$

where $C_{j_1 j_2 j_3 k_1 k_2 k_3}$ are the complex coefficients of f when f is transformed in complex variables ξ, η via the relations $x'_j = (\xi_j + i\,\eta_j)/\sqrt{2}$ and $y_j = i\,(\xi_j - i\,\eta_j)/\sqrt{2}$.

Suppose that the initial point of an orbit lies in the domain $\Delta_{\rho_0 R}$ for some positive ρ_0, and we ask the orbit to be confined inside a domain $\Delta_{\rho R}$ with $\rho_0<\rho$ for a finite time interval (escape time). In this case the inequality $\left| I'_j(t) - I'_j(0) \right| \le |t| \cdot \sup_{\Delta_{\rho R}} \left| \dot{I}'_j \right|$ for j=1,2,3, is true until the orbit eventually escapes from $\Delta_{\rho R}$. By $\sup_{\Delta_{\rho R}} \left| \dot{I}'_j \right|$ we denote the supremum of $\dot{I}'_j$ in the domain $\Delta_{\rho R}$. As explained in paper I, assuming that ρ is smaller than the half of the convergence radius of the remainder (3) we get:

$$\sup_{\Delta_{\rho R}} \left| \dot{I}'_j \right| < 2 \cdot \left\| \left[I'_j , Y^{(r)}_{r+1} \right] \right\|_{\rho R} = 2\rho^{r+1} \cdot \left\| \left[I'_j , Y^{(r)}_{r+1} \right] \right\|_{R} ,\quad j=1,2,3 \qquad (8),$$

where $Y^{(r)}_{r+1}$ is the first term of the remainder, which is a homogeneous polynomial of order r+1 and can be easily computed. Thus the escape time can be computed as:

$$\tau_r(\rho_0 , \rho) = \min_{j=1,2,3} \frac{R_j^2 (\rho^2 - \rho_0^2)}{4\rho^{r+1} \cdot \left\| \left[I'_j , Y^{(r)}_{r+1} \right] \right\|_R} \qquad (9).$$

In order to have the escape time as a function of ρ_0 we optimize $\tau_r(\rho_0, \rho)$ with respect to ρ and r. The r.h.s. of (9) has a maximum for $\rho = \rho_0 \sqrt{(r+1)/(r-1)}$. By putting this value in (9) we let r run from order 3 up to order 30 and we compute the maximum escape time T. For a general discussion and for making the results comparable to the ones in paper I we put $R_1 = R_2 = R_3 = 1$. The results can be seen in figure 1a where we plot the logarithm of the maximum escape time (logT) as a function of the logarithm of the radius of the initial domain ($\log\rho_0$), with solid line for the 3D case and with dashed line for the 2D case. In both cases the expansion of the normal form was done up to order 30. From figure 1a we see that for large domains the maximum time needed for an orbit to escape is small, while by taking an initially small domain around L_4 all the orbits can be confined inside it for very long time intervals.

A meaningful time interval for our system is the estimated age of the universe, which in our time units is 10^{10}. This value is marked in figure 1a by a horizontal line and it corresponds to $\log\rho_0$=-1.640, namely, $\rho_0 = 2.29\cdot10^{-2}$ for the 3D case and to $\log\rho_0$=-1.575, namely, $\rho_0 = 2.66\cdot10^{-2}$ for the 2D case. The largest value of ρ_0 in the planar case was found in paper I, when the expansions were made up to order 35 and it was $\rho_0 = 2.91\cdot10^{-2}$. In that case the effective stability region was big enough to include 4 real asteroids. We see that the radius of the effective stability region in the spatial case is about 14 percent smaller than the one computed for the planar case for r=30 and 21 percent smaller than the one we got for r=35. Thus it is evident that the estimated region in the 3D case is a realistic one since it is comparable to a region that includes real asteroids. This result improves significantly previous estimations of the effective stability region in the spatial restricted three body problem (Giorgilli et al., 1989 and Celletti and Giorgilli, 1991).

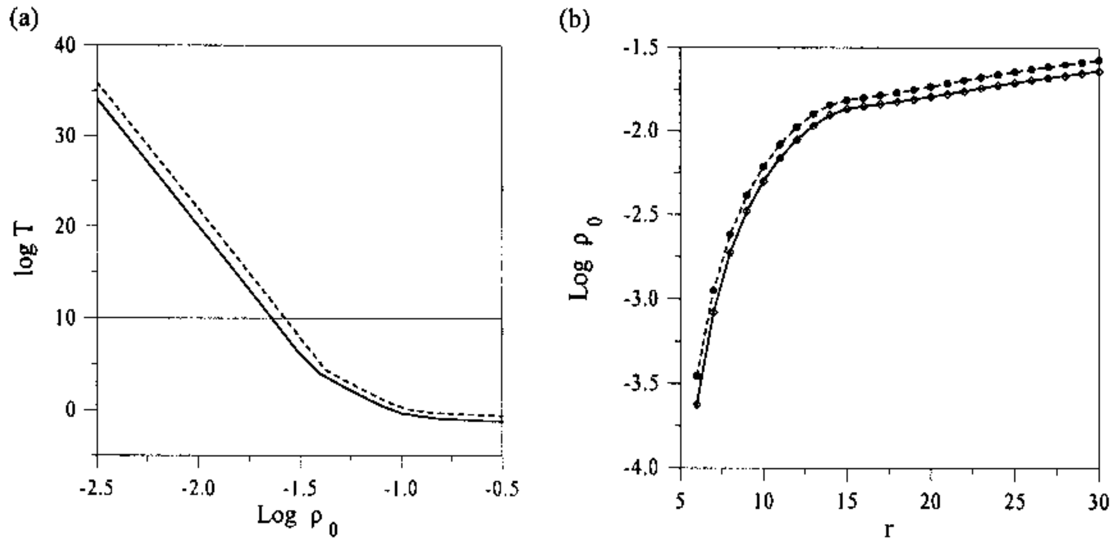

Figure 1. (a) The logarithm of the maximum escape time logT needed for orbits to leave domains around the point L_4, as a function of the logarithm of the domain radius $\log\rho_0$, in the 2D case (dashed line) and in the 3D case (solid line). In both cases the normal form has been computed up to order 30. A horizontal line marks the time corresponding to the age of the universe. (b) The logarithm of the radius $\log\rho_0$ of the effective stability region which ensures stability for time equal to the age of the universe, as a function of the order r of expansion of the normal form in the 2D case (black circles, dashed line) and in the 3D case (squares, solid line).

In figure 1b we see how the estimation of the radius of the stability region improves while the maximum order r, up to which the normal form is computed, increases both in the 2D case (dashed line) and in the 3D case (solid line). For every order the radius in the 3D case is smaller but close to the one computed for the 2D case. Up to order 15 the increment of the order r improves the estimation of the radius significantly. For r>15 the increment of the order by 1 leads to a big increment of the computational effort but to relatively small improvements of the radius. For instance the radius found in the 2D case for r=13 ($\log\rho_0$=-1.90) is almost equal to the one found for r=14 in the 3D case ($\log\rho_0$=-1.91), which is obtained by computing almost 18 times more coefficients than in the 2D case. As r increases things become even worse. For r=25 in the 2D case we get $\log\rho_0$=-1.64 which is the value we find for r=30 in the 3D case. It is remarkable that in the 3D case we use almost 100 times more coefficients. It becomes clear that in order to improve significantly the results in the 3D case one has to go to very high orders, which is impractical. On the other hand this can be done easier in the 2D case since we have a much smaller number of coefficients and the results for the radius are very close to the ones we get from the 3D case as seen in figure 1b.

According to Nekhoroshev's theory, the series arising from classical perturbation theory have an asymptotic character. This means that at some point one should reach an optimal value for the order of expansion, which gives the best possible result. As seen in figure 1b this optimal order is greater than r=30, thus the results obtained in this paper could be improved by computing the normal form up to higher orders.

4. Conclusions

We found a region of effective stability in the spatial restricted three body problem around the point L_4. We significantly improved older estimations since the size of this region is comparable to the size of the effective stability region found in paper I for the planar problem, which includes 4 real asteroids.

The results in the spatial and planar cases are similar for the same order of expansion. Since the computational effort for obtaining these results in the 3D case, becomes extremely larger than the effort in the 2D case, as the maximum order of the expansions grows, it is sufficient to improve the estimations in the 2D case.

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